Leray-Hopf and Continuity Properties for All Weak Solutions for the 3D Navier-Stokes Equations

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Abstract

In this note we prove that each weak solution for the 3D Navier-Stokes system satisfies Leray-Hopf property. Moreover, each weak solution is rightly continuous in the standard phase space H endowed with the strong convergence topology.

1 Introduction and Main Result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with rather smooth boundary $\Gamma = \partial\Omega$, and $[\tau, T]$ be a fixed time interval with $-\infty < \tau < T < +\infty$. We consider 3D Navier-Stokes system in $\Omega \times [\tau, T]$

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu \triangle y + (y \cdot \nabla) y = -\nabla p + f, & \text{div } y = 0, \\ y\big|_{\Gamma} = 0, & y\big|_{t=\tau} = y_\tau, \end{cases} \tag{1.1}$$

where $y(x,t)$ means the unknown velocity, $p(x,t)$ the unknown pressure, $f(x,t)$ the given exterior force, and $y_\tau(x)$ the given initial velocity with $t \in [\tau, T]$, $x \in \Omega$, $\nu > 0$ means the viscosity constant.

Throughout this note we consider generalized setting of Problem (1.1). For this purpose define the usual function spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \text{div } u = 0\},\ V_\sigma = \text{cl}_{(H_0^\sigma(\Omega))^3}\mathcal{V},\ \sigma \geq 0,$$

where cl_X denotes the closure in the space X. Set $H := V_0$, $V := V_1$. It is well known that each V_σ, $\sigma > 0$, is a separable Hilbert space and identifying H and its dual H^* we have $V_\sigma \subset H \subset V_\sigma^*$ with dense and

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compact embedding for each $\sigma > 0$. We denote by $(\cdot,\cdot)$, $\|\cdot\|$ and $((\cdot,\cdot))$, $\|\cdot\|_V$ the inner product and norm in H and V, respectively; $\langle\cdot,\cdot\rangle$ will denote pairing between V and V^* that coincides on $H \times V$ with the inner product $(\cdot,\cdot)$. Let H_w be the space H endowed with the weak topology. For $u, v, w \in V$ we put

$$b(u,v,w) = \int_\Omega \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

It is known that b is a trilinear continuous form on V and $b(u,v,v) = 0$, if $u, v \in V$. Furthermore, there exists a positive constant C such that

$$|b(u,v,w)| \le C\|u\|_V\|v\|_V\|w\|_V, \tag{1.2}$$

for each $u, v, w \in V$; see, for example, Sohr [17, Lemma V.1.2.1] and references therein.

Let $f \in L^2(\tau,T;V^*) + L^1(\tau,T;H)$ and $y_\tau \in H$. Recall that the function $y \in L^2(\tau,T;V)$ with $\frac{dy}{dt} \in L^1(\tau,T;V^*)$ is a *weak solution* of Problem (1.1) on $[\tau,T]$, if for all $v \in V$

$$\frac{d}{dt}(y,v) + \nu((y,v)) + b(y,y,v) = \langle f,v\rangle \tag{1.3}$$

in the sense of distributions, and

$$y(\tau) = y_\tau. \tag{1.4}$$

The weak solution y of Problem (1.1) on $[\tau,T]$ is called a *Leray-Hopf* solution of Problem (1.1) on $[\tau,T]$, if y satisfies the energy inequality:

$$V_\tau(y(t)) \le V_\tau(y(s)) \quad \text{for all } t \in [s,T], \text{ a.e. } s > \tau \text{ and } s = \tau, \tag{1.5}$$

where

$$V_\tau(y(\varsigma)) := \frac{1}{2}\|y(\varsigma)\|^2 + \nu\int_\tau^\varsigma \|y(\xi)\|_V^2 d\xi - \int_\tau^\varsigma \langle f(\xi), y(\xi)\rangle d\xi, \quad \varsigma \in [\tau,T]. \tag{1.6}$$

For each $f \in L^2(\tau,T;V^*) + L^1(\tau,T;H)$ and $y_\tau \in H$ there exists at least one Leray-Hopf solution of Problem (1.1); see, for example, Temam [18, Chapter III] and references therein. Moreover, $y \in C([\tau,T],H_w)$ and $\frac{dy}{dt} \in L^{\frac{4}{3}}(\tau,T;V^*) + L^1(\tau,T;H)$. If $f \in L^2(\tau,T;V^*)$, then, additionally, $\frac{dy}{dt} \in L^{\frac{4}{3}}(\tau,T;V^*)$. In particular, the initial condition (1.4) makes sense.

The following Theorem 1.1 implies that each weak solution of the 3D Navier-Stokes system is Leray-Hopf one and it is rightly strongly continuous in H at all the points $t \in [\tau,T)$. This theorem is the main result of this note.

Theorem 1.1. *Let $-\infty < \tau < T < +\infty$, $y_\tau \in H$, $f \in L^2(\tau,T;V^*) + L^1(\tau,T;H)$, and y be a weak solution of Problem (1.1) on $[\tau,T]$. Then the following statements hold:*

(a) *$y \in C([\tau,T],H_w)$ and the following energy inequality holds:*

$$V_\tau(y(t)) \le V_\tau(y(s)) \quad \textit{for all } t,s \in [\tau,T],\ t \ge s, \tag{1.7}$$

where V_τ is defined in formula (1.6);

(b) *for each* $t \in [\tau, T)$ *the following convergence holds:*

$$y(s) \to y(t) \text{ strongly in } H \text{ as } s \to t+;$$

(c) *the function* $t \to \|y(t)\|^2$ *is of bounded variation on* $[\tau, T]$.

Remark 1.2. Since a real function of bounded variation has no more than countable set of discontinuity points, then statement (a) of Theorem 1.1, weak continuity in Hilbert space H of each weak solution of Problem (1.1) on $[\tau, T]$, yield that each weak solution of the 3D Navier-Stokes system has no more than countable set of discontinuity points in the phase space H endowed with the strong convergence topology. Theorem 1.1 partially clarifies the results provided in Ball [1]; Balibrea et al. [2]; Barbu et al. [3]; Cao and Titi [4]; Chepyzhov and Vishik [5]; Cheskidov and Shvydkoy [6]; Kapustyan et al. [9, 10]; Kloeden et al. [13]; Sohr [17] and references therein.

2 Topological Properties of Solutions for Auxiliary Control Problem

Let $-\infty < \tau < T < +\infty$. We consider the following space of parameters:

$$\mathbb{U}_{\tau,T} := (L^2(\tau, T; V)) \times \left(L^2(\tau, T; V^*) + L^1(\tau, T; H)\right) \times H.$$

Each triple $(u, g, z_\tau) \in \mathbb{U}_{\tau,T}$ is called *admissible* for the following auxiliary control problem:

Problem **(C)** on $[\tau, T]$ with $(u, g, z_\tau) \in \mathbb{U}_{\tau,T}$: find $z \in L^2(\tau, T; V)$ with $\dfrac{dz}{dt} \in L^1(\tau, T; V^*)$ such that $z(\tau) = z_\tau$ and for all $v \in V$

$$\frac{d}{dt}(z, v) + \nu((z, v)) + b(u, z, v) = \langle g, v\rangle \tag{2.1}$$

in the sense of distributions; cf. Kapustyan et al. [9, 10]; Kasyanov et al. [11, 12]; Melnik and Toscano [14]; Zgurovsky et al. [19, Chapter 6].

As usual, let $A : V \to V^*$ be the linear operator associated with the bilinear form $((u, v)) = \langle Au, v\rangle$, $u, v \in V$. For $u, v \in V$ we denote by $B(u, v)$ the element of V^* defined by $\langle B(u, v), w\rangle = b(u, v, w)$, for all $w \in V$. Then Problem (C) on $[\tau, T]$ with $(u, g, z_\tau) \in \mathbb{U}_{\tau,T}$ can be rewritten as: find $z \in L^2(\tau, T; V)$ with $\dfrac{dz}{dt} \in L^1(\tau, T; V^*)$ such that

$$\frac{dz}{dt} + \nu Az + B(u, z) = g, \text{ in } V^*, \text{ and } z(\tau) = z_\tau. \tag{2.2}$$

The following theorem establishes the uniqueness properties for solutions of Problem (C).

Theorem 2.1. *Let* $-\infty < \tau < T < +\infty$ *and* $u \in L^2(\tau, T; V)$. *Then Problem (C) on* $[\tau, T]$ *with* $(u, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$ *has the unique solution* $z \equiv \bar{0}$.

We recall, that $\{w_1, w_2, \ldots\} \subset \mathcal{V}$ is the *special basis*, if $((w_j, v)) = \lambda_j(w_j, v)$ for each $v \in V$ and $j = 1, 2, \ldots$, where $0 < \lambda_1 \le \lambda_2 \le \ldots$ is the sequence of eigenvalues. Let P_m be the projection operator of H onto $H_m := \mathrm{span}\{w_1, \ldots, w_m\}$, that is $P_m v = \sum_{i=1}^m (v, w_i)w_i$ for each $v \in H$ and $m = 1, 2, \ldots$. Of course we may consider P_m as a projection operator that acts from V_σ onto H_m for each $\sigma > 0$ and, since $P_m^* = P_m$, we deduce that $\|P_m\|_{\mathcal{L}(V_\sigma^*; V_\sigma^*)} \le 1$. Note that $(w_j, v)_{V_\sigma} = \lambda_j^\sigma(w_j, v)$ for each $v \in V_\sigma$ and $j = 1, 2, \ldots$.

Proof of Theorem 2.1. Let $-\infty < \tau < T < +\infty$, $u \in L^2(\tau, T; V)$, and z be a solution of Problem (C) on $[\tau, T]$ with $(u, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$. Prove that $z \equiv \bar{0}$.

Let us fix an arbitrary $m = 1, 2, \ldots$. According to the definition of a solution for Problem (C) on $[\tau, T]$ with $(u, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$, the following equality holds:

$$\frac{1}{2}\frac{d}{dt}\|P_m z(t)\|^2 + \nu\|P_m z(t)\|_V^2 = b(u(t), P_m z(t), z(t)), \tag{2.3}$$

for a.e. $t \in (\tau, T)$. Since $b(u(t), P_m z(t), P_m z(t)) = 0$ for a.e. $t \in (\tau, T)$, then inequality (1.2) yields that

$$b(u(t), P_m z(t), z(t)) \leq C\|u(t)\|_V\|P_m z(t)\|_V\|z(t) - P_m z(t)\|_V,$$

for a.e. $t \in (\tau, T)$. Therefore, equality (2.3) imply the following inequality

$$\frac{1}{2}\frac{d}{dt}\|P_m z(t)\|^2 + \|P_m z(t)\|_V\left(\nu\|P_m z(t)\|_V - C\|u(t)\|_V\|z(t) - P_m z(t)\|_V\right) \leq 0, \tag{2.4}$$

for a.e. $t \in (\tau, T)$.

Let us set $\psi_m(t) := \|P_m z(t)\|_V\left(\nu\|P_m z(t)\|_V - C\|u(t)\|_V\|z(t) - P_m z(t)\|_V\right)$, for each $m = 1, 2, \ldots$ and a.e. $t \in (\tau, T)$. The following statements hold:

(i) $\psi_m \in L^1(\tau, T)$ for each $m = 1, 2, \ldots$;

(ii) $\psi_m(t) \leq \psi_{m+1}(t)$ for each $m = 1, 2, \ldots$ and a.e. $t \in (\tau, T)$;

(iii) $\psi_m(t) \to \nu\|z(t)\|_V^2$ as $m \to \infty$, for a.e. $t \in (\tau, T)$.

Indeed, statement (i) holds, because $u, z \in L^2(\tau, T; V)$ and $P_m z \in L^\infty(\tau, T; V)$ for each $m = 1, 2, \ldots$. Statement (ii) holds, because $\|P_m z(t)\|_V \leq \|P_{m+1} z(t)\|_V$ and $-\|z(t) - P_m z(t)\|_V \leq -\|z(t) - P_{m+1} z(t)\|_V$ for each $m = 1, 2, \ldots$ and a.e. $t \in (\tau, T)$. Statement (iii) holds, because $P_m z(t) \to z(t)$ strongly in V as $m \to \infty$, for a.e. $t \in (\tau, T)$.

Since $\|z(\cdot)\|_V^2 \in L^1(\tau, T)$, then statements (i)–(iii) and Lebesgue's monotone convergence theorem yield

$$\lim_{m\to\infty}\int_\tau^t \psi_m(s)ds = \int_\tau^t \lim_{m\to\infty}\psi_m(s)ds = \int_\tau^t \|z(s)\|_V^2 ds, \tag{2.5}$$

for each $t \in [\tau, T]$. Inequality (2.4) implies

$$\frac{1}{2}\|P_m z(t)\|^2 + \nu\int_\tau^t \psi_m(s)ds = \int_\tau^t \frac{1}{2}\frac{d}{dt}\|P_m z(t)\|^2 + \nu\int_\tau^t \psi_m(s)ds \leq 0, \tag{2.6}$$

for each $m = 1, 2, \ldots$ and $t \in [\tau, T]$. We note that the equality in (2.6) holds, because $z(\tau) = \bar{0}$.

Equality (2.5) and inequality (2.6) yield that

$$\frac{1}{2}\|z(t)\|^2 + \nu\int_\tau^t \|z(s)\|_V^2 ds \leq 0,$$

for a.e. $t \in (\tau, T)$, because $P_m z(t) \to z(t)$ strongly in H for a.e. $t \in (\tau, T)$. Thus, $z(t) = \bar{0}$ for a.e. $t \in (\tau, T)$. Since $z \in C([\tau, T]; V^*)$, then $z \equiv \bar{0}$, that is, Problem (C) on $[\tau, T]$ with $(u, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$ has the unique solution $z \equiv \bar{0}$. $\square$

The following theorem establishes sufficient conditions for the existence of an unique solution for Problem (C). This is the main result of this section.

Theorem 2.2. *Let $-\infty < \tau < T < +\infty$, $y_\tau \in H$, $f \in L^2(\tau, T; V^*) + L^1(\tau, T; H)$, and y be a weak solution of Problem (1.1) on $[\tau, T]$. Then $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$ and Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$ has the unique solution $z = y$. Moreover, y satisfies inequality (1.5).*

Before the proof of Theorem 2.2 we remark that $AC([\tau, T]; H_m)$, $m = 1, 2, \dots$, will denote the family of absolutely continuous functions acting from $[\tau, T]$ into H_m, $m = 1, 2, \dots$.

Proof of Theorem 2.2. Prove that $z = y$ is the unique solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$. Indeed, y is the solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$, because y is a weak solution of Problem (1.1) on $[\tau, T]$. Uniqueness holds, because if z is a solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$, then $z - y \equiv \bar{0}$ is the unique solution of Problem (C) on $[\tau, T]$ with $(y, \bar{0}, \bar{0}) \in \mathbb{U}_{\tau,T}$ (see Theorem 2.1).

The rest of the proof establishes that y satisfies inequality (1.5). We note that y can be obtained via standard Galerkin arguments, that is, if $y_m \in AC([\tau, T]; H_m)$ with $\frac{d}{dt} y_m \in L^1(\tau, T; H_m)$, $m = 1, 2, \dots$, is the approximate solution such that

$$\frac{dy_m}{dt} + \nu A y_m + P_m B(y, y_m) = P_m f, \text{ in } H_m, \quad y_m(\tau) = P_m y(\tau), \tag{2.7}$$

then the following statements hold:

(i) y_m satisfy the following energy equality:

$$\begin{aligned} &\frac{1}{2}\|y_m(t_1)\|^2 + \nu \int_s^{t_1} \|y_m(\xi)\|_V^2 d\xi - \int_s^{t_1} \langle f(\xi), y_m(\xi)\rangle d\xi \\ &\qquad = \frac{1}{2}\|y_m(t_2)\|^2 + \nu \int_s^{t_2} \|y_m(\xi)\|_V^2 d\xi - \int_s^{t_2} \langle f(\xi), y_m(\xi)\rangle d\xi, \end{aligned} \tag{2.8}$$

for each $t_1, t_2 \in [\tau, T]$, for each $m = 1, 2, \dots$;

(ii) there exists a subsequence $\{y_{m_k}\}_{k=1,2,\dots} \subseteq \{y_m\}_{m=1,2,\dots}$ such that the following convergence (as $m \to \infty$) hold:

$(\mathrm{ii})_1$ $y_{m_k} \to y$ weakly in $L^2(\tau, T; V)$;

$(\mathrm{ii})_2$ $y_{m_k} \to y$ weakly star in $L^\infty(\tau, T; H)$;

$(\mathrm{ii})_3$ $P_{m_k} B(u, y_{m_k}) \to B(u, y)$ weakly in $L^2(\tau, T; V^*_{\frac{3}{2}})$;

$(\mathrm{ii})_4$ $P_{m_k} f \to f$ strongly in $L^2(\tau, T; V^*) + L^1(\tau, T; H)$;

$(\mathrm{ii})_5$ $\frac{dy_{m_k}}{dt} \to \frac{dy}{dt}$ weakly in $L^2(\tau, T; V^*_{\frac{3}{2}}) + L^1(\tau, T; H)$.

Indeed, convergence $(\mathrm{ii})_1$ and $(\mathrm{ii})_2$ follow from (2.8) (see also Temam [18, Remark III.3.1, pp. 264, 282]) and Banach-Alaoglu theorem. Since there exists $C_1 > 0$ such that $|b(u, v, w)| \le C\|u\|_V \|w\|_V \|v\|_V^{\frac{1}{2}} \|v\|^{\frac{1}{2}}$, for each $u, v, w \in V$ (see, for example, Sohr [17, Lemma V.1.2.1]), then $(\mathrm{ii})_1$, $(\mathrm{ii})_2$ and Banach-Alaoglu

theorem imply $(\mathrm{ii})_3$. Convergence $(\mathrm{ii})_4$ holds, because of the basic properties of the projection operators $\{P_m\}_{m=1,2,\ldots}$. Convergence $(\mathrm{ii})_5$ directly follows from $(\mathrm{ii})_3$, $(\mathrm{ii})_4$ and (2.7). We note that we may not to pass to a subsequence in $(\mathrm{ii})_1$–$(\mathrm{ii})_5$, because $z = y$ is the unique solution of Problem (C) on $[\tau, T]$ with $(y, f, y_\tau) \in \mathbb{U}_{\tau,T}$.

Moreover, there exists a subsequence $\{y_{k_j}\}_{j=1,2,\ldots} \subseteq \{y_{m_k}\}_{k=1,2,\ldots}$ such that

$$y_{k_j}(t) \to y(t) \text{ strongly in } H \text{ for a.e. } t \in (\tau, T) \text{ and } t = \tau, \quad j \to \infty. \tag{2.9}$$

Indeed, according to (2.7), (2.8) and $(\mathrm{ii})_3$, the sequence $\{y_{m_k} - F_{m_k}\}_{k=1,2,\ldots}$, where $F_{m_k}(t) := \int_\tau^t P_{m_k} f(s)ds$, $m = 1, 2, \ldots$, $t \in [\tau, T]$, is bounded in a reflexive Banach space $W_{\tau,T} := \{w \in L^2(\tau, T; V) : \frac{d}{dt}w \in L^2(\tau, T; V^*_{\frac{3}{2}})\}$. Compactness lemma yields that $W_{\tau,T} \subset L^2(\tau, T; H)$ with compact embedding. Therefore, $(\mathrm{ii})_1$–$(\mathrm{ii})_5$ imply that $y_{m_k} \to y$ strongly in $L^2(\tau, T; H)$ as $m \to \infty$. Thus, there exists a subsequence $\{y_{k_j}\}_{j=1,2,\ldots} \subseteq \{y_{m_k}\}_{k=1,2,\ldots}$ such that (2.9) holds.

Due to convergence $(\mathrm{ii})_1$–$(\mathrm{ii})_5$ and (2.9), if we pass to the limit in (2.8) as $m_{k_j} \to \infty$, then we obtain that y satisfies the inequality

$$\frac{1}{2}\|y(t)\|^2 + \nu \int_s^t \|y(\xi)\|_V^2 d\xi - \int_s^t \langle f(\xi), y(\xi)\rangle d\xi \le \frac{1}{2}\|y(\tau)\|^2, \tag{2.10}$$

for a.e. $t \in (s, T)$, a.e. $s \in (\tau, T)$ and $s = \tau$.

Since $y \in L^\infty(\tau, T; H) \cap C([\tau, T]; V^*)$ and $H \subset V^*$ with continuous embedding, then $y \in C([\tau, T]; H_w)$. Thus, equality (2.10) yields

$$\frac{1}{2}\|y(t)\|^2 + \nu \int_s^t \|y(\xi)\|_V^2 d\xi - \int_s^t \langle f(\xi), y(\xi)\rangle d\xi \le \frac{1}{2}\|y(\tau)\|^2,$$

for each $t \in [\tau, T]$, a.e. $s \in (\tau, T)$ and $s = \tau$. Therefore, y satisfies inequality (1.5). □

3 Proof Theorem 1.1

In this section we establish the proof of Theorem 1.1. Let Π_{t_1,t_2} be the restriction operator to the finite time subinterval $[t_1, t_2] \subseteq [\tau, T]$; Chepyzhov and Vishik [5].

Proof of Theorem 1.1. Let $-\infty < \tau < T < +\infty$, $y_\tau \in H$, $f \in L^2(\tau, T; V^*) + L^1(\tau, T; H)$, and y be a weak solution of Problem (1.1) on $[\tau, T]$.

Let us prove statement (a). Fix an arbitrary $s \in [\tau, T)$. Since $(\Pi_{s,T}y, \Pi_{s,T}f, y(s)) \in \mathbb{U}_{s,T}$, then Theorem 2.2 yields that $\Pi_{s,T}y \in L^\infty(s, T; H)$ and it satisfies the following inequality:

$$V_\tau(y(t)) \le V_\tau(y(s)) \quad \text{for all } t \in [s, T],$$

where V_τ is defined in formula (1.6). Since $s \in [\tau, T)$ be an arbitrary, then statement (a) holds.

Let us prove statement (b). Statement (a) yields

$$\frac{1}{2}\|y(t)\|^2 + \nu \int_s^t \|y(\xi)\|_V^2 d\xi - \int_s^t \langle f(\xi), y(\xi)\rangle d\xi \le \frac{1}{2}\|y(s)\|^2, \tag{3.1}$$

for each $t \in [s, T]$, for each $s \in [\tau, T)$. In particular, $\limsup_{t \to s+} \|y(t)\| \leq \|y(s)\|$ for all $s \in [\tau, T)$, and

$$y(t) \to y(s) \text{ strongly in } H \text{ as } t \to s+ \text{ for each } s \in [\tau, T), \tag{3.2}$$

because $y \in C([\tau, T]; H_w)$.

Let us prove statement (c). Since $y \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ and $f \in L^2(\tau, T; V^*) + L^1(\tau, T; H)$, then statements (a) and (b) imply that the mapping $t \to \|y(t)\|^2$ is of bounded variation on $[\tau, T]$. $\square$

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